Solution to IBM Ponder This July 2016

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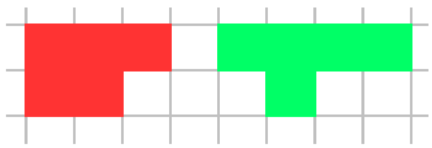
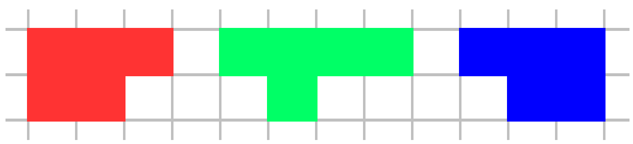
1 Answer

Below is one of the solutions that can tile every $4^N \times 4^N$ board with a missing square, if rotation and flipping are all allowed.

We'll call the set of these two pentominoes "Answer".



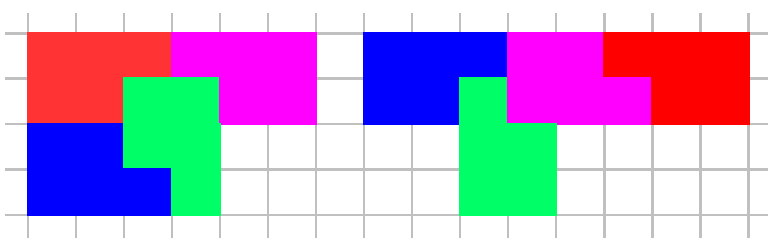
If flipping is not allowed, we can use three pentominoes below.



2 Proof For Flipping-OK Situation

2.1 Forming A Larger Pentomino

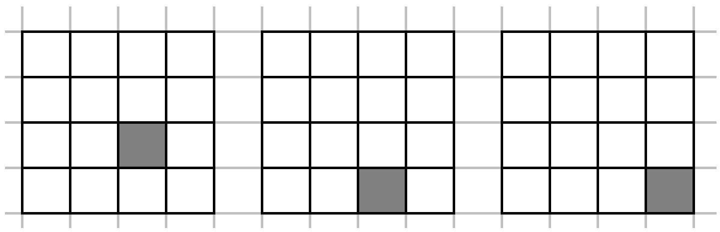
We can use pentominoes in Answer to form a double-sized pentomino in Answer:



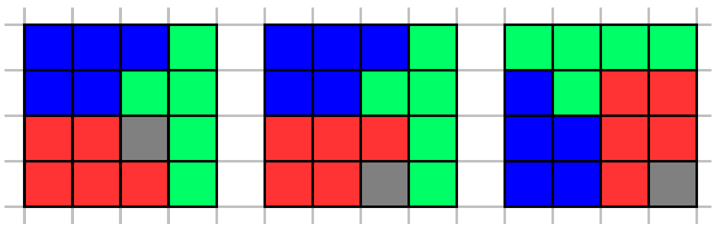
With this method, for every integer $i \geq 0$ we can form a pentomino that is similar to some pentomino in Answer and the ratio of similarity is 2^i. We call that "i-th large pentomino". For example, the picture above shows two 1st large pentominoes.

2.2 Proof For 4×4 Case

We show how to use Answer to tile 4×4 board with any missing square. If allowing rotation and flipping, we only need to consider the following three situations:



Answer is :



2.3 The Final Proof

We need to proof

Theorem 1. *every $4^N \times 4^N$ board with a missing square can be tiled by pentominos in Answer.*

We use induction : we've proved $N = 1$ case in section 2.2.

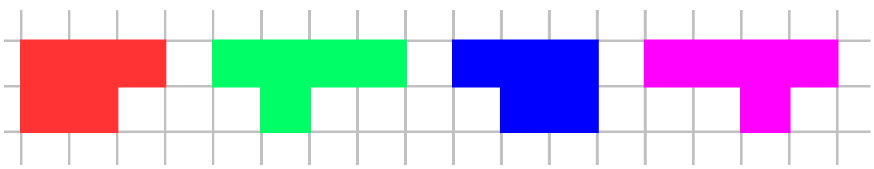
Assume that theorem 1 is correct for $N = k$. Consider $N = k+1$, number the rows and columns $0 \sim 4^N - 1$, let the missing cell be (x, y). Divide the board into 16 $4^k \times 4^k$ subboards.

First we cover the 15 subboards which don't contain the missing cell. It's always possible to use k-th large pentominoes in Answer to cover them as it's equivalent to a problem where $N = 1$. As every k-th large pentominoes in Answer can be formed by original pentominoes in Answer, it's possible to cover the 15 subboards above.

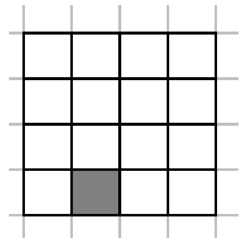
Then we need to deal with the remaining subboard. It's a $4^k \times 4^k$ board with a missing cell, according to induction it's possible to cover it using pentominoes in Answer. □

3 If Flipping Is Banned

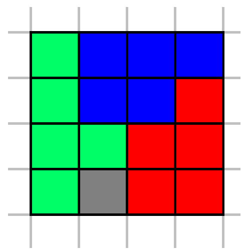
Obviously it's achievable by 4 pentominoes :



Let's show that the pink one is not needed. Now flipping is banned and we need to consider one more situation than in section 2.2:



which has solution



And obviously k-th large pentominoes are still constructable. Similar to section 2.3, we can prove that

Theorem 2. *every $4^N \times 4^N$ board with a missing square can be tiled by pentominoes in the above picture. It's not allowed to flip a pentomino, and we don't need to use the pink pentomino.*