Solution To Uyhip 2016 Nov

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Note : I used python programs to find the pattern(though the proof is my original). Please add me to the solvers list only if you think using programs is legal.

We prove that

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k+j}=\frac{1}{j\binom{n+j}{j}} \tag{1}$$

by induction of n.

When $n=0$, $LHS=\frac{1}{j}=RHS$.

Suppose for all $j>0$, equation

$$\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{(-1)^k}{k+j}=\frac{1}{j\binom{n-1+j}{j}} \tag{2}$$

holds.

Then,

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k+j}\\
=\quad&\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{(-1)^k}{k+j}+\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{(-1)^k}{k+j}\\
=\quad&\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{(-1)^k}{k+j}+\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{(-1)^{k+1}}{k+j+1}\\
=\quad&\frac{1}{j\binom{n-1+j}{j}}-\frac{1}{(j+1)\binom{n+j}{j+1}}\\
=\quad&\frac{(j+1)\binom{n+j}{j+1}-j\binom{n-1+j}{j}}{j(j+1)\binom{n-1+j}{j}\binom{n+j}{j+1}}\\
=\quad&\frac{\frac{n(n-1+j)!}{(n-1)!j!}}{j(j+1)\binom{n-1+j}{j}\binom{n+j}{j+1}}\\
=\quad&\frac{n(n-1)!(j+1)!}{j(j+1)(n+j)!}\\
=\quad&\frac{1}{j\binom{n+j}{j}}
\end{aligned}$$

That is, if the equation is right for $n-1$, it's also right for n.

Q.E.D.